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STATISTICS ON WREATH PRODUCTS AND GENERALIZED BINOMIAL-STIRLING NUMBERS

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ABSTRACT

Various statistics on wreath products are defined via canonical words, "colored" right to left minima and "colored" descents. It is shown that refined counts with respect to these statistics have nice recurrence formulas of binomial-Stirling type. These extended Stirling numbers determine (via matrix inversion) dual systems, which are also shown to have combinatorial realizations within the wreath product. The above setting also gives rise to a MacMahon-type equi-distribution theorem over subsets with prescribed statistics.

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1. Introduction

This paper was motivated by [15, 16], where we studied a variety of natural statistics on the symmetric group S_n which generalized the *length*, the *major* and other statistics. In particular, new statistics based on canonical presentations by the Coxeter generators were introduced. Then the various Stirling numbers were obtained as cardinalities of certain subsets of S_n defined via these statistics. For example, the Stirling numbers of the second kind are cardinalities of subsets of permutations with prescribed number of left-to-right minima and descents. Refinements of the classical MacMahon-type equi-distribution theorems [10] — in the spirit of the results of Foata–Schützenberger, Garsia–Gessel etc. — were deduced.

In this paper the group of permutations S_n is replaced by the wreath product $C_a \wr S_n$, whose elements are called *"colored permutations"*. Here C_a is the cyclic group with a elements. We study canonical presentations in wreath products and introduce statistics counting the number of "long" and of "short" factors in these presentations. These numbers essentially count the number of certain right to left minima in colored permutations. It is shown that enumeration of elements in wreath products with respect to these (and to these and descent) statistics have nice recurrence formulas of binomial-Stirling type. In particular, we present a wreath product extension of Stirling numbers of first and second kinds [18], interpret these numbers in the wreath product, and prove a MacMahon-type equi-distribution theorem over subsets with prescribed statistics.

Fix four integers $a, d, r, \ell \in Z$ and let $g(n,k) = g_{a,d,r,\ell}(n,k)$ be the numbers determined by the following recurrence:

$g(0,0) = 1$ and

$$g(n,k) = (an + dk - r) \cdot g(n-1,k) + \ell \cdot g(n-1,k-1), \tag{1}$$

and $g(n,k) = 0$ if $k < 0$ or $n < k$.

The numbers $g_{a,d,r,\ell}(n,k)$ combine and generalize the binomial coefficients and the Stirling numbers; see Section 8. For example, $g_{1,0,1,1}(n,k)$ are the *signless Stirling numbers of the first kind*, $g_{0,1,0,1}(n,k)$ are the *Stirling numbers of the second kind*, and $g_{0,0,-1,1}(n,k)$ are the *binomial coefficients*.

For a positive integer a and a subset $L \subseteq \{0, \ldots, a-1\}$ of cardinality ℓ let

$$A_L(n,k) := \{\sigma \in C_a \wr S_n \mid \overleftarrow{\min}_L (\sigma) = k\}$$

and

$$B_L(n,k) := \{\sigma \in C_a \wr S_n \mid des_L(\sigma) = \overleftarrow{\min}_L (\sigma) = k\},$$

where $\overleftarrow{\min}_L(\sigma)$ is the number of L-colored right to left minima (see Definition 4.1.2) and $des_L(\sigma)$ is the number of descents with respect to the L-order (see Definitions 4.5 and 4.7). Then

THEOREM 1.1 (see Corollary 5.2 and Theorem 6.6):

$$g_{a,0,\ell,\ell} = \#A_L(n,k)$$

and

$$g_{0,a,\ell-a,\ell}(n,k) = \#B_L(n,k).$$

These two systems are essentially dual. This is

THEOREM 1.2 (See Theorem 9.4): *For every positive integer a, N, and every subset $L \subseteq \{0,\ldots,a-1\}$ of size ℓ, let $s_{L,N}$ be the $N \times N$ matrix whose entries are given by*

$$s_{L,N}(n,k) := \frac{(-1)^{n-k}}{\ell^n} \cdot \#A_L(n,k) \quad (0 \le k,n \le N)$$

and $S_{L,N}$ be the $N \times N$ matrix whose entries are

$$S_{L,N}(n,k) := \frac{1}{\ell^n} \cdot \#B_L(n,k) \quad (0 \le k,n \le N).$$

Then

$$S_{L,N}^{-1} = s_{L,N}.$$

To prove this theorem we apply a general decomposition and inversion theorems for linear recurrences; see Theorems 8.4 and 8.8 below. These theorems are closely related to results of Milne and followers [11, 12, 13].

In Section 7 we apply the above setting to show that the length function and the flag major index are equi-distribution over subsets of $B_n = C_2 \wr S_n$ with prescribed colored right-to-left minima; see Corollaries 7.5 and 7.6. This result is a type B-analogue of a recent theorem of Foata and Han for the symmetric group [7, (1.5)] and refines a recent result of Haglund, Loehr and Remmel [9, Theorem 4.5].

The rest of the paper is organized as follows.

Basic facts about wreath products are given in Sections 2 and 3. In Section 4, statistics on $C_a \wr S_n$ based on canonical words and on "colored" orders are introduced. Generalized Stirling numbers are interpreted combinatorially in Sections 5, 6 and 9, and are formally studied in Section 8 and Appendix 2. The main equi-distribution theorem, Theorem 7.3, is given in Section 7.

Remark: It was recently brought to our attention that certain refinements of the classical MacMahon equi-distribution theorem, which were proved in [15], were already studied by Björner and Wachs [5]. In particular, [15, Theorem 1.9(1)] follows from [5, Example 5.3]. See also [14].

2. Preliminaries

THE WREATH PRODUCT $C_a \wr S_n$. Let G be a group. Recall that the elements of the wreath product $G \wr S_n$ are of the form $\sigma = ((x_1, \ldots, x_n), p)$ where $x_i \in G$ and $p \in S_n$; multiplication is given by

$$((x_1, \ldots, x_n), p)((y_1, \ldots, y_n), q) = ((x_1 y_{p^{-1}(1)}, \ldots, x_n y_{p^{-1}(n)}), pq).$$

Let A be the set $A := G \times \{1, \ldots, n\} \equiv \{xj \mid x \in G,\ 1 \le j \le n\}$. We identify $((x_1, \ldots, x_n), p)$ with the function $((x_1, \ldots, x_n), p) \equiv f\colon A \to A$, given by

$$f\colon tj \to (t x_{p(j)}) p(j)$$

for all $t \in G$ and $1 \le j \le n$. When G is Abelian one verifies easily that if, also, $g \equiv ((y_1, \ldots, y_n), q)$ then $fg \equiv ((x_1, \ldots, x_n), p)((y_1, \ldots, y_n), q)$. This justifies the above identification. We therefore represent the element $\sigma = ((x_1, \ldots, x_n), p) \in G \wr S_n$ by the n-tuple $[x_{p(1)} p(1), \ldots, x_{p(n)} p(n)]$:

$$\sigma = ((x_1, \ldots, x_n), p) \equiv [x_{p(1)} p(1), \ldots, x_{p(n)} p(n)] = [\sigma(1), \ldots, \sigma(n)],$$

and we denote $|\sigma| := p$. Note that if $\sigma = [y_1 j_1, \ldots, y_n j_n]$ where $y_i \in G$ and $1 \le j_i \le n$, then $|\sigma| = p = [j_1, \ldots, j_n]$. Let $z_i \in G$, $p \in S_n$ and let $\sigma = [z_1 p(1), \ldots, z_n p(n)]$; then $\sigma^{-1} = [z^{-1}_{p^{-1}(1)} p^{-1}(1), \ldots, z^{-1}_{p^{-1}(n)} p^{-1}(n)]$.

In this paper we consider the wreath products $C_a \wr S_n$, where C_a is the (multiplicative) cyclic group of order a: $\alpha := e^{2\pi i/a}$, and

$$C_a := \{\alpha^t \mid 0 \le t \le a-1\}.$$

The elements of $C_a \wr S_n$ are identified with "a-colored" permutations, namely those permutations σ of the set $A = C_a \times \{1, \ldots, n\}$ satisfying

$$\sigma(\beta j) = \beta \sigma(j), \quad \beta \in C_a, \quad 1 \le j \le n.$$

We write $\sigma = [\sigma(1), \ldots, \sigma(n)]$. For each j, $\sigma(j) = \alpha^{t_j} \cdot |\sigma(j)|$ and has *color* α^{t_j}; it is "colorless" if $t_j = 0$.

Cycle decomposition. Let $\sigma = ((x_1, \ldots, x_n), p) \in C_a \wr S_n$. The cycle decomposition of p induces the corresponding decomposition of σ: If $p = p_1 \cdots p_r$ is the cycle decomposition of p, and $p_i = (b_1^{(i)}, \ldots, b_m^{(i)})$ (in the ordinary cycle notation for S_n), for each $1 \le i \le r$ let

$$y_j^{(i)} := \begin{cases} x_j, & \text{if } j \in \{b_1^{(i)}, \ldots, b_m^{(i)}\}; \\ 1, & \text{otherwise.} \end{cases}$$

Then $\sigma^{(i)} := ((y_1^{(i)}, \ldots, y_n^{(i)}), p_i)$ are the corresponding cycles of σ, and $\sigma = \sigma^{(1)} \cdots \sigma^{(r)}$ is the *cycle decomposition* of σ. The product $x_{b_1^{(i)}} \cdots x_{b_m^{(i)}} \in C_a$ is uniquely determined (since C_a is Abelian), and is called the *color of that cycle* of $p = |\sigma|$.

Generators and length. Let $s_i = (i, i+1) \in S_n$, $i = 1, \ldots, n-1$, denote the Coxeter generators of $S_n \subset C_a \wr S_n$. In addition, $s_0 \in C_a \wr S_n$ is the element given by

$$s_0 = ((\alpha, 1, \ldots, 1), 1) \equiv [\alpha, 2, 3, \ldots, n]$$

where $\alpha = e^{2\pi i/a}$.

The following easy fact is well known.

Fact 2.1: Let $\sigma = [b_1, \ldots, b_n] \in C_a \wr S_n$.

1. $\sigma s_0 = [\alpha b_1, b_2, \ldots, b_n]$.
2. Let $1 \le i \le n-1$; then $\sigma s_i = [b_1, \ldots, b_{i+1}, b_i, \ldots, b_n]$.

The set $S = \{s_0, s_1, \ldots, s_{n-1}\} \subseteq C_a \wr S_n$ generates $C_a \wr S_n$ (this follows, for example, from Proposition 3.1).

The *length* of an element $\sigma \in C_a \wr S_n$, denoted $\ell(\sigma)$, is the minimum length of an expression of σ as a product of elements in the above generating set S.

3. Canonical presentation in wreath products

Consider the following subsets of elements in $C_a \wr S_n$. First, let $R_0 = C_a$. Given $1 \le j \le n-1$, let

$$R_j^0 := \{1, s_j, s_j s_{j-1}, \ldots, s_j \cdots s_1\}.$$

For $1 \le t \le a-1$ let

$$R_j^t := \{s_j \cdots s_0^t, s_j \cdots s_0^t s_1, s_j \cdots s_0^t s_1 s_2, \ldots, s_j \cdots s_0^t s_1 \cdots s_j\}$$

and

$$R_j := \bigcup_{t=0}^{a-1} R_j^t.$$

Note that $|R_j| = a \cdot (j+1)$, hence

$$\prod_{j=0}^{n-1} |R_j| = a^n \cdot n! = |C_a \wr S_n|.$$

PROPOSITION 3.1: *Every element $\sigma \in C_a \wr S_n$ has a unique presentation*

$$\sigma = w_0 \cdots w_{n-1}$$

where, for every $0 \le j \le n-1$, $w_j \in R_j$.

Proof: By induction on n and by Fact 2.1. Recall that every element $\sigma \in C_a \wr S_n$ may be interpreted as a colored permutation $[\sigma(1), \dots, \sigma(n)]$. It follows from this interpretation that every element $\sigma \in C_a \wr S_n$ is obtained in a unique way by inserting colored n (namely $e^{2\pi i t/a} n$ for some $0 \le t \le a-1$) into a colored permutation $\bar{\sigma} \in C_a \wr S_{n-1}$. Now, if $\sigma(j) = e^{2\pi i t/a} n$ and $t = 0$ then $\sigma = \bar{\sigma} w_{n-1}$, where

$$w_{n-1} = \begin{cases} 1, & \text{if } j = n, \\ s_{n-1} \cdots s_j, & \text{if } j < n, \end{cases} \in R^0_{n-1}.$$

If $\sigma(j) = e^{2\pi i t/a} n$ and $0 < t \le a-1$ then $\sigma = \bar{\sigma} w_{n-1}$, where

$$w_{n-1} = s_{n-1} \cdots s_0^t \cdots s_{j-1} \in R^t_{n-1}.$$

This proves "existence". Uniqueness now follows by a standard counting argument. ∎

Definition 3.2: Call the above presentation $\sigma = w_0 \cdots w_{n-1}$ in Proposition 3.1 the canonical presentation — or the canonical word — of $\sigma = w_0 \cdots w_{n-1}$.

PROPOSITION 3.3: *Write the above canonical word explicitly: $\sigma = w_0 \cdots w_{n-1} = s_{i_1} \cdots s_{i_r}$. Then r is the minimum length of an expression of σ as a product of elements in $S = \{s_0, s_1, \dots, s_{n-1}\}$, i.e. the length of σ is $\ell(\sigma) = r$.*

For a proof see, e.g., [4, Ch. 3.3].

COROLLARY 3.4: *Let $\sigma = w_0 \cdots w_{n-1}$ be the canonical word of $\sigma \in C_a \wr S_n$, then $\ell(\sigma) = \ell(w_0) + \cdots + \ell(w_{n-1})$. In particular, if $\bar{\sigma} \in C_a \wr S_{n-1}$ and $r \in R_{n-1}$ then $\ell(\bar{\sigma} r) = \ell(\bar{\sigma}) + \ell(r)$.*

4. Statistics on colored permutations

In this section we introduce various statistics on $C_a \wr S_n$ based on canonical words, on right-to-left-minima, and on certain descent sets Des_L.

4.1 RIGHT TO LEFT MINIMA. Recall from Section 2 the notation $\alpha := e^{2\pi i/a}$ and $|\sigma|$ (for every $\sigma \in C_a \wr S_n$).

Definition 4.1:

1. Let $p = [j_1, \ldots, j_n] \in S_n$. Define $\overleftarrow{\mathrm{Min}}(p) \subseteq \{1, \ldots, n\}$ as follows:

$$\overleftarrow{\mathrm{Min}}(p) = \{j_i \mid j_i \text{ is a r.t.l.min in } [j_1, \ldots, j_n]\}.$$

   Here and below r.t.l.min stands for right to left minimum.

2. Let $L \subseteq \{1, \ldots, a-1\}$. Let $\sigma \in C_a \wr S_n$ be a colored permutation, and write $\sigma = [b_1, \ldots, b_n]$. Define $\overleftarrow{\mathrm{Min}}_L(\sigma) \subseteq \{1, \ldots, n\}$ as follows:

$$\overleftarrow{\mathrm{Min}}_L(\sigma) = \{|b_i| \mid |b_i| \text{is a r.t.l.min in } |\sigma|, \text{ and } b_i = \alpha^u |b_i| \text{ for some } u \in L\}.$$

   Finally, denote $\overleftarrow{\min}_L(\sigma) = |\overleftarrow{\mathrm{Min}}_L(\sigma)|$.

For example, let $\sigma = [\alpha 3, \alpha^3 5, 1, \alpha^2 2, \alpha 4] \in C_4 \wr S_5$; then $|\sigma| = [3,5,1,2,4]$ and $\overleftarrow{\mathrm{Min}}_{\{0,1,2,3\}}(\sigma) = \{1,2,4\}$, $\overleftarrow{\mathrm{Min}}_{\{1,2\}}(\sigma) = \{2,4\}$, $\overleftarrow{\mathrm{Min}}_{\{0,3\}}(\sigma) = \{1\}$, and $\overleftarrow{\mathrm{Min}}_{\{0,1,2\}}(\sigma) = \{1,2,4\}$.

PROPOSITION 4.2:

1. *Let $p = [j_1, \ldots, j_n] \in S_n$, let $v_0 = 1$ and let $p = v_0 v_1 \cdots v_{n-1}$ be its canonical presentation. Then j_i is a r.t.l.min in $[j_1, \ldots, j_n]$ if and only if $v_{j_i - 1} = 1$.*
1. *Let $\sigma \in C_a \wr S_n$ and $\sigma = w_0 \cdots w_{n-1}$ ($\forall i \; w_i \in R_i$) be its canonical word. Also, let $|\sigma| = v_1 \cdots v_{n-1}$ be the canonical presentation of $|\sigma|$. For each $0 \le u \le a-1$ and $1 \le j \le n-1$ denote*

$$r_{u,j} := \begin{cases} s_j \cdots s_0^u \cdots s_j \in R_j & \text{if } u \neq 0, \\ 1 & \text{if } u = 0. \end{cases}$$

   Then $v_i = 1$ if and only if $w_i = r_{u,i}$ for some u.

3. *Let $L \subseteq \{0, \ldots, a-1\}$. Then*

$$\overleftarrow{\mathrm{Min}}_L(\sigma) = \{i+1 \mid 0 \le i \le n-1 \quad \text{and} \quad \exists u \in L, \; w_i = r_{u,i}\}.$$

Proof: The proof is standard (by induction on n) and is left to the reader.

Example 4.3: As before, let $\sigma = [\alpha 3, \alpha^3 5, 1, \alpha^2 2, \alpha 4] \in C_4 \wr S_5$. First, $|\sigma| = p = [3,5,1,2,4] = (s_2 s_1)(s_4 s_3 s_2) = v_0 \cdots v_4$, where $v_0 = v_1 = v_3 = 1$, $v_2 = s_2 s_1$ and $v_4 = s_4 s_3 s_2$. Clearly, the elements which are the l.t.r.min of p are 1, 2 and 3, and indeed $v_{1-1} = v_{2-1} = v_{4-1} = 1$.

Next, the canonical presentation of σ is $\sigma = w_0 \cdots w_{n-1}$, where $w_0 = 1$, $w_1 = s_1 s_0^2 s_1, s_2 s_1 s_0$, $w_3 = s_3 s_2 s_1 s_0 s_1 s_2 s_3$ and $w_4 = s_4 s_3 s_2 s_1 s_0^3 s_1$. The elements $r_{u,i}$ here are $r_{0,0} = w_0 = 1$, $r_{2,1} = w_1$ and $r_{1,3} = w_3$. Now let, for example, $L = \{0,1,2,3\}$; then $\overleftarrow{\mathrm{Min}}_L(\sigma) = \{1,2,4\} = \{0+1, 1+1, 3+1\}$.

COROLLARY 4.4: *Let $\bar\sigma \in C_a \wr S_{n-1}$ and $r = w_{n-1} \in R_{n-1}$ (hence $\sigma \in C_a \wr S_n$). Let*

$$K_L(r) = K_L(w_{n-1}) := \begin{cases} \{n-1\} & \text{if } \exists u \in L \; w_{n-1} = r_{u,n-1}, \\ \emptyset & \text{otherwise.} \end{cases} \tag{2}$$

Then

$$\overleftarrow{\mathrm{Min}}_L(\sigma) = \overleftarrow{\mathrm{Min}}_L(\bar\sigma) \cup K_L(w_{n-1}), \text{ a disjoint union.} \tag{3}$$

4.2 THE ORDER $<_L$ AND THE L-DESCENT SET. Notice that $C_a \wr S_n$ is identified with the permutations σ of the set $\{\alpha^v j \mid 0 \le v \le a-1, 1 \le j \le n\} \cup \{0\}$ where, by definition, $\sigma(0) = 0$ and $\sigma(\alpha^v j) = \alpha^v \sigma(j)$.

Definition 4.5: A subset $L \subseteq \{0, \ldots, a-1\}$ determines a linear order $<_L$ on $\{\alpha^v j \mid 0 \le v \le a-1, 0 \le j \le n\} \cup \{0\}$ as follows:

Let $U = \{0, \ldots, a-1\} \setminus L$ be the complement of L in $\{0, \ldots, a-1\}$.

If $v \in L$ then $\alpha^v j <_L 0$ for every $1 \le j \le n$. If $v \in U$ then $\alpha^v j >_L 0$ for every $1 \le j \le n$.

For $v, u \in L$ (not necessarily distinct) and $i \ne j \in \{1, \ldots, n\}$, $\alpha^v i <_L \alpha^u j$ if and only if $i > j$ ("reverse order").

For $v, u \in U$ (not necessarily distinct) and $i \ne j \in \{1, \ldots, n\}$, $\alpha^v i <_L \alpha^u j$ if and only if $i < j$.

Then, for each $1 \le j \le n$, order each subset $\{\alpha^v j \mid v \in L\}$ (and each subset $\{\alpha^v j \mid v \in U\}$) in an arbitrary linear order.

This yields a linear order $<_L$ on the set $\{\alpha^v j \mid 0 \le v \le a-1, 0 \le j \le n\} \cup \{0\}$.

For example, let $a = 4$ and $L = \{2,3\}$; then $U = \{0,1\}$. We can choose the following order:

$$\alpha^2 n <_L \alpha^3 n <_L \alpha^2(n-1) <_L \alpha^3(n-1) <_L \cdots <_L \alpha^2 <_L \alpha^3 <_L 0,$$
$$0 <_L \alpha <_L 1 <_L \alpha 2 <_L 2 <_L \cdots <_L \alpha(n-1) <_L (n-1) <_L \alpha n <_L n.$$

The following is an obvious property of this order.

FACT 4.6: Let $\bar{\sigma} = [\bar{\sigma}(1), \ldots, \bar{\sigma}(n-1)] \in C_a \wr S_{n-1}$ and let $0 \le v \le a-1$.
If $v \in L$ then $\alpha^v n <_L \bar{\sigma}(1), \ldots, \bar{\sigma}(n-1)$;
if $v \in U$ then $\alpha^v n >_L \bar{\sigma}(1), \ldots, \bar{\sigma}(n-1)$.

Definition 4.7: The L-descent set of $\sigma \in C_a \wr S_n$ is

$$Des_L(\sigma) := \{0 \le i \le n-1 \mid \sigma(i) >_L \sigma(i+1)\}.$$

The L-descent number is

$$des_L(\sigma) := |\,Des_L(\sigma)|.$$

If L consists of one element $u \in \{0, \ldots, a-1\}$ then we denote $<_u$, Des_u, des_u.

The following notion is the natural analogue of the standard descent sets of Weyl and Coxeter groups.

Definition 4.8: For $\sigma \in C_a \wr S_n$ let the *standard descent set* be

$$Des(\sigma) := \{0 \le i \le n-1 | \; \ell(\sigma s_i) < \ell(\sigma)\}.$$

It should be noted that the u-descent set, Des_u, defined above, may also be interpreted via the generators.

PROPOSITION 1.1: *For every $\sigma \in C_a \wr S_n$ and every $0 \le u \le a-1$,*

$$Des_u(\sigma) = \{0 \le i \le n-1 | \; \ell(v_u^{-1}\sigma s_i) > \ell(v_u^{-1}\sigma)\},$$

where $v_u := ((\alpha^u, \ldots, \alpha^u), 1) = [\alpha^u 1, \alpha^u 2, \ldots, \alpha^u n]$.

Proof: The proof is given in Appendix 1 (Section 10).

Example 4.10:

(1) $L = \{0, \ldots, a-1\}$. By definition,

$$Des_{\{0,\ldots,a-1\}}(\sigma) = Des(|\sigma|) = \{0 \le i \le n-1 | \; |\sigma(i)| > |\sigma(i+1)|\},$$

the standard descent set of $|\sigma|$.

(2) $L = \emptyset$. $Des_\emptyset(\sigma)$ is the complement of the standard descent set of $|\sigma|$ (the ascent set of $|\sigma|$).

(3) $L = \{1, \ldots, a-1\}$. By Proposition 4.9, since v_0 is the identity element,

$$\begin{aligned} Des_{\{1,\ldots,a-1\}}(\sigma) &= \{0, \ldots, n-1\} \setminus Des_0(\sigma) \\ &= \{0 \le i \le n-1 | \; \ell(\sigma s_i) < \ell(\sigma)\}, \end{aligned}$$

the standard descent set of σ.

(4) $L = \{0, \ldots, a-2\}$. v_{a-1} is the longest element in $C_a \wr S_n$ and $Des_{\{0,\ldots,a-2\}}$ is the complement of the standard descent set, namely, the ascent set of σ.

LEMMA 4.11: *Let $L \subseteq \{0, \ldots, a-1\}$. Then, for any $\sigma \in C_a \wr S_n$,*

$$des_L(\sigma) \geq \overleftarrow{\min}_L (\sigma).$$

Proof: Let $\overleftarrow{\mathrm{Min}}_L (\sigma) = \{i_1, \ldots, i_k\}$, and show that for each $1 \leq j \leq k-1$, $\sigma(i_j) >_L \sigma(i_{j+1})$. Indeed, each $\sigma(i_t) = \alpha^{v_t}|\sigma(i_t)|$, $v_t \in L$, and $|\sigma(i_t)|$ is a r.t.l.min of $|\sigma|$. Therefore $|\sigma(i_j)| < |\sigma(i_{j+1})|$, so

$$\sigma(i_j) = \alpha^{v_j}|\sigma(i_j)| >_L \alpha^{v_{j+1}}|\sigma(i_{j+1})| = \sigma(i_{j+1}),$$

as was claimed. By the transitivity of the linear order $>_L$, there must be an L-descent of σ between these two indices i_j and i_{j+1}. This contributes (at least) $k-1$ L-descents to $Des_L(\sigma)$. By definition, $\sigma(0) = 0 >_L \sigma(i_1)$, and this contributes at least one more L-descent of σ. ∎

Note: Here, we have to allow $0 \in Des_L(\sigma)$.

5. "Colored" Stirling numbers of the first kind

In this section we point to connections between statistics on colored permutations, defined above, and certain generalized Stirling numbers of the first kind.

PROPOSITION 5.1: *Let $L \subseteq \{0, \ldots, a-1\}$, $|L| = r$. Then*

$$\sum_{\sigma \in C_a \wr S_n} q^{\overleftarrow{\min}_L(\sigma)} = (rq + a - r)(rq + 2a - r) \cdots (rq + na - r).$$

Proof: By Corollary 4.4 it suffices to show that for every n,

$$\sum_{w_{n-1} \in R_{n-1}} q^{|K_L(w_{n-1})|} = rq + na - r.$$

Indeed, by definition (2) (in Corollary 4.4)

$$\sum_{w_{n-1} \in R_{n-1}} q^{|K_L(w_{n-1})|} = rq + |R_{n-1}| - r = rq + na - r. \quad ∎$$

COROLLARY 5.2: *Let $r = |L|$ as above, and denote*

$$g_L(n,k) := \#\{\sigma \in C_a \wr S_n \,|\overleftarrow{\min}_L(\sigma) = k\}.$$

Then $g_L(n,k)$ satisfies the following recurrence:

$$g_L(n,k) = (an - r) \cdot g_L(n-1,k) + r \cdot g_L(n-1,k-1).$$

Thus, by Equation (1), $g_L(n,k) = g_{a,0,r,r}(n,k)$, so

$$g_{a,0,r,r}(n,k) = \#\{\sigma \in C_a \wr S_n \,|\overleftarrow{\min}_L(\sigma) = k\}.$$

Proof: By Proposition 5.1

$$\sum_k g_L(n,k)q^k = \sum_{\sigma \in C_a \wr S_n} q^{\overleftarrow{\min}_L(\sigma)} = (rq + a - r)(rq + 2a - r)\cdots(rq + na - r).$$

Thus

$$\begin{aligned}\sum_k g_L(n,k)q^k &= (rq + na - r)\sum_k g_L(n-1,k)q^k \\ &= (na - r)\sum_k g_L(n-1,k)q^k + \sum_k r \cdot g_L(n-1,k-1)q^k,\end{aligned}$$

and the proof follows. ∎

Note: When $a = |L| = r = 1$, $g_L(n,k)$ are the signless Stirling numbers of the first kind. In Section 8 we study similar but more general such recurrences.

Recall from Section 2 that the cycles of $\sigma \in C_a \wr S_n$ are "colored" by elements of C_a.

Definition 5.3: Given $L \subseteq \{1, \ldots, n\}$ and $\sigma \in C_a \wr S_n$, we say that a cycle of σ is L-colored if its color belongs to L.

COROLLARY 5.4: *The number of elements $\sigma \in C_a \wr S_n$ with exactly k r.t.l.min of $|\sigma|$ which are L-colored, $g_L(n,k)$, is also the number of elements $\sigma \in C_a \wr S_n$ with exactly k cycles which are L-colored.*

Proof: The proof is a natural extension of [17, p. 17]. The following notion will be used in the proof. Let $\sigma = ((x_1, \ldots, x_n), p) \in C_a \wr S_n$, and let $\gamma = (b_1, \ldots, b_m)$ be a cycle of $p = |\sigma|$. Assume w.l.o.g. that the last element b_m is minimal; then the color of b_m, x_{b_m}, will be called *the right-color* of the cycle γ. A cycle is *right L-colored*, for $L \subseteq \{0, \ldots, a-1\}$, if its right-color belongs to $\{\alpha^u |\, u \in L\}$.

Let $G'_L(n,k)$ denote the set of elements $\sigma \in C_a \wr S_n$ with exactly k cycles which are right L-colored and

$$G_L(n,k) := \{\sigma \in C_a \wr S_n \,|\overleftarrow{\min}_L(\sigma) = k\}.$$

We first construct a bijection

$$G'_L(n,k) \longleftrightarrow G_L(n,k).$$

Given $\sigma' = ((x_1,\ldots,x_n),p') \in G'_L(n,k)$, reorder the cycles in $p' = |\sigma'|$ such that each cycle in $|\sigma'|$ is written with its smallest element last (i.e. rightmost), and the cycles are written in increasing order of their smallest element. By assumption, exactly k of these smallest elements are L-colored. Let p be the permutation obtained from p' by erasing the parentheses of the cycles, and let $\sigma = ((x_1,\ldots,x_n),p)$. Clearly, in $p = |\sigma|$, those smallest elements are now r.t.l.min, and in σ they have the same colors as in σ', namely exactly k of these r.t.l.min are L-colored. Thus $\sigma \in G_L(n,k)$. That correspondence can be reversed by parenthesizing $p \in S_n$ according to its r.t.l.min, therefore the above is a bijection.

Let $G''_L(n,k)$ denote the set of elements $\sigma \in C_a \wr S_n$ with exactly k cycles which are L-colored. There is a rather obvious bijection

$$G''_L(n,k) \longleftrightarrow G'_L(n,k)$$

as follows. Given $\sigma'' = ((x_1,\ldots,x_n),p'') \in G''_L(n,k)$, let $(b_1,\ldots,b_m)$ be a cycle of p'' with b_m minimal, then replace x_{b_m} by $x_{b_1}\cdots x_{b_m}$. Do it to each cycle. This clearly maps $G''_L(n,k) \longrightarrow G'_L(n,k)$, with an obvious inverse map. This completes the proof. ∎

6. "Colored" Stirling numbers of the second kind

In this section we prove the second part of Theorem 1.1 (Theorem 6.6 below).

Throughout this section we assume that $L \subseteq \{0,1,\ldots,a-1\}$, with the corresponding linear order $<_L$ as above.

LEMMA 6.1: *Let $\sigma = w_0 \cdots w_{n-1}$ (canonical presentation), $\bar\sigma = w_0 \cdots w_{n-2}$, so $\sigma = \bar\sigma w_{n-1}$. Then $des_L(\sigma) \geq_L des_L(\bar\sigma)$.*

Proof: Recall that σ is obtained from $\bar\sigma$ by inserting some $\alpha^v n$ into $\bar\sigma$. Thus, for certain $b_1,\ldots,b_{n-1} \in C_a \cdot \{1,\ldots,n-1\}$ and $1 \leq t \leq n-1$,

$$\bar\sigma = [b_1,\ldots,b_{n-1},n] \quad \text{and} \quad \sigma = [b_1,\ldots,b_t,\alpha^v n, b_{t+1},\ldots,b_{n-1}].$$

Since the L-order is linear, if $b_t >_L b_{t+1}$ then either $b_t >_L \alpha^v n$ or (and/or) $\alpha^v n >_L b_{t+1}$, which implies the proof. ∎

LEMMA 6.2: *With the notation of the previous Lemma,*

1. *if* $\sigma(n) = \alpha^v n$ *for some* $v \in L$ *then* $\overleftarrow{\mathrm{Min}}_L(\sigma) = \overleftarrow{\mathrm{Min}}_L(\bar{\sigma}) \cup \{n\}$, *hence* $\overleftarrow{\min}_L(\sigma) = \overleftarrow{\min}_L(\bar{\sigma}) + 1$;
2. *if* $\sigma(n) \neq \alpha^v n$ *for any* $v \in L$ *then* $\overleftarrow{\min}_L(\sigma) = \overleftarrow{\min}_L(\bar{\sigma})$.

Proof: The lemma is an immediate consequence of Corollary 4.4. ∎

The following is a key observation here.

LEMMA 6.3: *Let* $\sigma = \bar{\sigma} w_{n-1}$ *as above, and assume* $des_L(\sigma) = \overleftarrow{\min}_L(\sigma) = k$.

1. *If* $\sigma(n) = \alpha^v n$ *for some* $v \in L$ *then* $des_L(\bar{\sigma}) = \overleftarrow{\min}_L(\bar{\sigma}) = k - 1$.
2. *If* $\sigma(n) \neq \alpha^v n$ *for any* $v \in L$ *then* $des_L(\bar{\sigma}) = \overleftarrow{\min}_L(\bar{\sigma}) = k$.

Proof: 1. Assume $\sigma(n) = \alpha^v n$, $v \in L$. By Lemma 6.2.1, $\overleftarrow{\min}_L(\bar{\sigma}) = \overleftarrow{\min}_L(\sigma) - 1$. Clearly in that case $Des_L(\sigma) = Des_L(\bar{\sigma}) \cup \{n-1\}$, hence also $des_L(\bar{\sigma}) = k-1$.

2. If $\sigma(n) \neq \alpha^v n$ for any $v \in L$ then, by Lemma 6.2.2, $\overleftarrow{\min}_L(\sigma) = \overleftarrow{\min}_L(\bar{\sigma})$. Therefore by Lemmas 6.1 and 4.11,

$$k = des_L(\sigma) \geq des_L(\bar{\sigma}) \geq \overleftarrow{\min}_L(\bar{\sigma}) = \overleftarrow{\min}_L(\sigma) = k,$$

forcing equality. Thus $des_L(\bar{\sigma}) = \overleftarrow{\min}_L(\bar{\sigma}) = k$. ∎

LEMMA 6.4: *Let* $\bar{\sigma} \in C_a \wr S_{n-1}$, $\bar{\sigma} = [\bar{\sigma}(1), \ldots, \bar{\sigma}(n-1)]$. *Assume that* $des_L(\bar{\sigma}) = \overleftarrow{\min}_L(\bar{\sigma}) = k$ *and let* $Des_L(\bar{\sigma}) = \{i_1, \ldots, i_k\}$.

1. *If* $v \notin L$, *then there are exactly* $k+1$ *elements* $\sigma \in C_a \wr S_n$ *such that* $\sigma = \bar{\sigma} w_{n-1}$ *for some* $w_{n-1} \in R_{n-1}$ *and* $des_L(\sigma) = \overleftarrow{\min}_L(\sigma) = k$.
2. *If* $v \in L$, *then there are exactly* k *such* σ*'s "over"* $\bar{\sigma}$ *satisfying* $des_L(\sigma) = \overleftarrow{\min}_L(\sigma) = k$.

Proof: Fix some $b_n = \alpha^v n$ and insert it into $\bar{\sigma}$ to obtain $\sigma = \bar{\sigma} w_{n-1}$.

1. $v \notin L$, hence $\bar{\sigma}(1), \ldots, \bar{\sigma}(n-1) <_L b_n$. If b_n is inserted immediately to the right of some $\bar{\sigma}(i_t)$ $(1 \leq t \leq k)$ or in the last (n-th) position, then $des_L(\sigma) = k$. Also, by Lemma 6.2.2, $\overleftarrow{\min}_L(\sigma) = \overleftarrow{\min}_L(\bar{\sigma}) = k$. Conversely, if $des_L(\sigma) = k$ then b_n was inserted into one of these $k+1$ positions.

2. $v \in L$, hence $b_n <_L, 0, \bar{\sigma}(1), \ldots, \bar{\sigma}(n-1)$. If b_n is inserted immediately to the right of some $\bar{\sigma}(i_t)$ $(1 \leq t \leq k)$ then $des_L(\sigma) = k$. Also, in this case b_n is

not inserted in the last position; by Lemma 6.2.2, $\overleftarrow{\min}_L(\sigma) = \overleftarrow{\min}_L(\bar{\sigma}) = k$. Conversely, if $des_L(\sigma) = k$ then b_n was inserted into one of these k positions.

Note that if $i_1 \neq 0$ and b_n is inserted in the first (left) position then 0 is an additional u-descent of σ, since $0 >_L b_n$. ∎

Definition 6.5: Let $f_L(0,0) = 1$ and define

$$f_L(n,k) = \#\{\sigma \in C_a \wr S_n \mid des_L(\sigma) = \overleftarrow{\min}_L(\sigma) = k\}.$$

THEOREM 6.6: *Let $\ell = |L|$. Then $f_L(n,k)$ satisfies the following recurrence:*

$$f_L(n,k) = (ak + a - \ell)\cdot f_L(n-1,k) + \ell \cdot f_L(n-1,k-1).$$

Thus $f_L(n,k) = g_{0,a,\ell-a,\ell}(n,k)$, so

$$g_{0,a,\ell-a,\ell}(n,k) = \#\{\sigma \in C_a \wr S_n \mid des_L(\sigma) = \overleftarrow{\min}_L(\sigma) = k\}.$$

Proof: Let

$$B_L(n,k) := \{\sigma \in C_a \wr S_n \mid des_L(\sigma) = \overleftarrow{\min}_L(\sigma) = k\},$$

so $f_L(n,k) = \#B_L(n,k)$,

$$C_L(n,k) := \{\sigma = \bar{\sigma} w_{n-1} \in B_L(n,k) \mid \bar{\sigma} \in B_L(n-1,k-1)\},$$

and

$$D_L(n,k) := \{\sigma = \bar{\sigma} w_{n-1} \in B_L(n,k) \mid \bar{\sigma} \in B_L(n-1,k)\}.$$

By definition, $C_L(n,k) \cap D_L(n,k) = \emptyset$. By Lemma 6.3,

$$B_L(n,k) = C_L(n,k) \cup D_L(n,k).$$

The proof will follow, once we show that

1. $|C_L(n,k)| = \ell \cdot |B_L(n-1,k-1)|$ and
2. $|D_L(n,k)| = (ak + a - \ell)\cdot |B_L(n-1,k)|$.

1. By Lemma 6.3, all elements in $C_L(n,k)$ which are obtained from an element $\bar{\sigma} \in B_L(n-1,k-1)$ by inserting a colored n are obtained by inserting an L-colored n at the last position: $\sigma = [\bar{\sigma}, \alpha^v n]$, $v \in L$. This proves 1.

2. Let $\bar{\sigma} \in B_L(n-1,k)$: $des_L(\bar{\sigma}) = \overleftarrow{\min}_L(\bar{\sigma}) = k$ and insert $\alpha^v n$ into $\bar{\sigma}$ to obtain a permutation $\sigma = \bar{\sigma} w_{n-1} \in B_L(n,k)$. If $v \notin L$ then, by Lemma 6.4.1, there are exactly $k+1$ such permutations $\sigma \in B_L(n,k)$. Since there are $a-\ell$ such v's, we get $(a-\ell)(k+1)$ σ's. Similarly, Lemma 6.4.2 implies k such σ's when $v \in L$, namely a total of ℓk σ's. Together, this yields exactly $(a-\ell)(k+1) + \ell k = ak + a - \ell$ σ's in $B_L(n,k)$ "over" each $\bar{\sigma} \in B_L(n-1,k)$. This proves 2. ∎

Remark 6.7: Letting $a = \ell = 1$, $f_L(n,k)$ are the classical Stirling numbers of the second kind.

7. Equi-distribution in $B_n = C_2 \wr S_n$

In this section we study the case of $B_n = C_2 \wr S_n$, namely $a = 2$. We prove here an equi-distribution theorem between the length parameter $\ell(\sigma)$ and the flag-major index; see Definition 7.2 below.

Here $L \subseteq \{0,1\}$ determines $<_L$. In the case $L = \{1\}$ the natural order is preserved, and it is reversed when $L = \{0\}$:

$$-n <_1 -(n-1) <_1 \cdots <_1 -1 <_1 0 <_1 1 <_1 \cdots <_1 n \quad \text{and}$$
$$n <_0 n-1 <_0 \cdots <_0 1 <_0 0 <_0 -1 <_0 \cdots <_0 -n.$$

These orders define the corresponding $\overleftarrow{\mathrm{Min}}_0$ and $\overleftarrow{\mathrm{Min}}_1$ sets; see Definition 4.1. In this section we show that the length function and the flag major index are equi-distribution over subsets of $B_n = C_2 \wr S_n$ with prescribed $\overleftarrow{\mathrm{Min}}_0$ and $\overleftarrow{\mathrm{Min}}_1$ sets; see Corollary 7.5 below. This result is a type B-analogue of a recent theorem of Foata and Han for the symmetric group [7, (1.5)] and refines a recent result of Haglund, Loehr and Remmel [9, Theorem 4.5].

THEOREM 7.1: *For every positive integer n*

$$\sum_{\sigma \in B_n} \prod_{i \in \overleftarrow{\mathrm{Min}}_0(\sigma)} x_i \cdot \prod_{i \in \overleftarrow{\mathrm{Min}}_1(\sigma)} t_i \cdot q^{\ell(\sigma)} =$$
$$(x_1 + qt_1)(x_2 + q + q^2 + q^3 t_2) \cdots (x_n + q + q^2 + \cdots + q^{2n-1} t_n).$$

Proof: By induction on n. Obviously, the theorem holds for $n = 1$. By Proposition 3.1, the l.h.s. equals

$$\sum_{\bar\sigma \in B_{n-1}} \sum_{r \in R_{n-1}} q^{\ell(\bar\sigma r)} \cdot \prod_{i \in \overleftarrow{\mathrm{Min}}_0(\bar\sigma r)} x_i \cdot \prod_{i \in \overleftarrow{\mathrm{Min}}_1(\bar\sigma r)} t_i = Q.$$

By Remark 3.4 and Corollary 4.4, Q equals

$$\left[\sum_{\bar\sigma \in B_{n-1}} \prod_{i \in \overleftarrow{\mathrm{Min}}_0(\bar\sigma)} x_i \cdot \prod_{i \in \overleftarrow{\mathrm{Min}}_1(\bar\sigma)} t_i \cdot q^{\ell(\bar\sigma)}\right] \cdot \left[\sum_{r \in R_{n-1}} \prod_{i \in K_0(r)} x_i \cdot \prod_{i \in K_1(r)} t_i \cdot q^{\ell(r)}\right].$$

Thus, by induction, it suffices to show that

$$\left[\sum_{r \in R_{n-1}} \prod_{i \in K_0(r)} x_i \cdot \prod_{i \in K_1(r)} t_i \cdot q^{\ell(r)}\right] = x_n + q + q^2 + \cdots + q^{2n-1} t_n.$$

Recall that in the case of B_n, $R_{n-1} = R^0_{n-1} \cup R^1_{n-1}$, where

$$R^0_{n-1} := \{1, s_{n-1}, s_{n-1}s_{j-2}, \ldots, s_{n-1}\cdots s_1\} \quad \text{and}$$
$$R^1_{n-1} := \{s_{n-1}\cdots s_0, s_{n-1}\cdots s_0 s_1, s_{n-1}\cdots s_0 s_1 s_2, \ldots, s_{n-1}\cdots s_0 s_1 \cdots s_{n-1}\}.$$

The only $r = w_{n-1}$ in R_{n-1} with $K_0(r) \neq \emptyset$ is $r = 1$, hence the contribution of x_n. Similarly, the only $r = w_{n-1}$ in R_{n-1} with $K_1(r) \neq \emptyset$ is $r = s_{n-1}\cdots s_0 \cdots s_{n-1}$ — of length $2n-1$, hence the contribution of $q^{2n-1}t_n$. This also explains the other summands q, q^2, etc.

This implies the proof. ∎

Definition 7.2: For $\sigma \in C_2 \wr S_n = B_n$ it is natural to consider the following descent set:

$$Des_A(\sigma) := \{0 \leq i \leq n-1 \mid \sigma(i) > \sigma(i+1)\}$$

and the following major index:

$$maj_A(\sigma) := \sum_{i \in Des_A(\sigma)} i.$$

Let

$$neg(\sigma) := \#\{i \mid \sigma(i) < 0\}$$

and define the flag major index as

$$fmaj(\sigma) := 2 \cdot maj_A(\sigma) + neg(\sigma).$$

The flag major index was introduced in [3] in order to extend the MacMahon classical equi-distribution theorem to B_n. For a unified definition of the classical major index and the flag-major index as a length of a distinguished canonical expression, see [3, Theorem 3.1]. The flag-major index has many other combinatorial and algebraic properties which are shared with the classical major index on S_n; see, for example, [1, 2, 9] and references therein.

The following theorem is a flag-major index analogue of Theorem 7.1.

THEOREM 7.3: *For every positive integer n*

$$\sum_{\sigma \in B_n} \prod_{i \in \overleftarrow{\text{Min}}_0(\sigma)} x_i \cdot \prod_{i \in \overleftarrow{\text{Min}}_1(\sigma)} t_i \cdot q^{fmaj(\sigma)} =$$
$$(x_1 + qt_1)(x_2 + q + q^2 + q^3 t_2) \cdots (x_n + q + q^2 + \cdots + q^{2n-1}t_n).$$

To prove this theorem we need the following lemma.

LEMMA 7.4: *For every $\bar\sigma \in B_{n-1}$*

$$\sum_{r\in R_{n-1}} q^{fmaj(\bar\sigma r)} = q^{fmaj(\bar\sigma)} \cdot (1+q+\cdots+q^{2n-1}).$$

Proof: By the definition of *fmaj* (Definition 7.2),

$$\begin{aligned}\sum_{r\in R_{n-1}} q^{fmaj(\bar\sigma r)} &= \sum_{r\in R_{n-1}} q^{2maj_A(\bar\sigma r)+neg(\bar\sigma r)}\\ &= \sum_{r\in R^0_{n-1}} q^{2maj_A(\bar\sigma r)+neg(\bar\sigma r)} + \sum_{r\in R^1_{n-1}} q^{2maj_A(\bar\sigma r)+neg(\bar\sigma r)}\\ &= \sum_{r\in R^0_{n-1}} q^{2maj_A(\bar\sigma r)+neg(\bar\sigma)} + \sum_{r\in R^1_{n-1}} q^{2maj_A(\bar\sigma r)+neg(\bar\sigma)+1}\\ &= q^{neg(\bar\sigma)}\cdot\left[\sum_{r\in R^0_{n-1}} q^{2maj_A(\bar\sigma r)} + q\sum_{r\in R^1_{n-1}} q^{2maj_A(\bar\sigma r)}\right].\end{aligned}$$

By a theorem of Garsia and Gessel [8, Theorem 3.1],

$$\sum_{r\in R^0_{n-1}} q^{2maj_A(\bar\sigma r)} = \sum_{r\in R^1_{n-1}} q^{2maj_A(\bar\sigma r)} = q^{2maj_A(\bar\sigma)}\cdot(1+q^2+\cdots+q^{2(n-1)}),$$

completing the proof of the lemma. ∎

Proof of Theorem 7.3.: Again, by induction on n. Obviously, the theorem holds for $n=1$.

Recall the definition of $r_{u,n-1}$ from Proposition 4.2. Then for every $\bar\sigma\in B_{n-1}$,

$$fmaj(\bar\sigma r_{1,n-1}) = fmaj(\bar\sigma)+2n-1 \quad fmaj(\bar\sigma\cdot r_{0,n-1}) = fmaj(\bar\sigma). \tag{4}$$

Combining (4) with Lemma 7.4 implies

$$\sum_{r\in R_{n-1}\setminus\{r_{0,n-1},r_{1,n-1}\}} q^{fmaj(\bar\sigma r)} = q^{fmaj(\bar\sigma)}\cdot(q+\cdots+q^{2n-2}). \tag{5}$$

Clearly, the l.h.s. in the theorem equals

$$\sum_{\bar\sigma\in B_{n-1}}\sum_{r\in R_{n-1}\setminus\{r_{0,n-1},r_{1,n-1}\}}\prod_{i\in\overleftarrow{\mathrm{Min}}_0(\bar\sigma r)} x_i\cdot\prod_{i\in\overleftarrow{\mathrm{Min}}_1(\bar\sigma r)} t_i\cdot q^{fmaj(\bar\sigma r)}$$
$$+\sum_{\bar\sigma\in B_{n-1}}\sum_{r\in\{r_{0,n-1},r_{1,n-1}\}}\prod_{i\in\overleftarrow{\mathrm{Min}}_0(\bar\sigma r)} x_i\cdot\prod_{i\in\overleftarrow{\mathrm{Min}}_1(\bar\sigma r)} t_i\cdot q^{fmaj(\bar\sigma r)}.$$

By Corollary 4.4 and (5), the first sum equals

$$\sum_{\bar\sigma\in B_{n-1}}\sum_{r\in R_{n-1}\setminus\{r_{0,n-1},r_{1,n-1}\}} q^{fmaj(\bar\sigma r)}\cdot\prod_{i\in\overleftarrow{\mathrm{Min}}_0(\bar\sigma)} x_i\cdot\prod_{i\in\overleftarrow{\mathrm{Min}}_1(\bar\sigma)} t_i$$
$$=\prod_{i\in\overleftarrow{\mathrm{Min}}_0(\bar\sigma)} x_i\cdot\prod_{i\in\overleftarrow{\mathrm{Min}}_1(\bar\sigma)} t_i\cdot q^{fmaj(\bar\sigma)}(q+\cdots+q^{n-2}).$$

By Corollary 4.4 and (4), the second sum equals

$$\prod_{i\in\overleftarrow{\mathrm{Min}}_0(\bar\sigma)} x_i\cdot\prod_{i\in\overleftarrow{\mathrm{Min}}_1(\bar\sigma)} t_i\cdot q^{fmaj(\bar\sigma)}(x_n+t_nq^{2n-1}),$$

completing the proof. ∎

We deduce

Corollary 7.5: *For every positive integer n*

$$\sum_{\sigma\in B_n}\prod_{i\in\overleftarrow{\mathrm{Min}}_0(\sigma)} x_i\cdot\prod_{i\in\overleftarrow{\mathrm{Min}}_1(\sigma)} t_i\cdot q^{\ell(\sigma)}=\sum_{\sigma\in B_n}\prod_{i\in\overleftarrow{\mathrm{Min}}_0(\sigma)} x_i\cdot\prod_{i\in\overleftarrow{\mathrm{Min}}_1(\sigma)} t_i\cdot q^{fmaj(\sigma)}.$$

Equivalently, for every positive integer n and every pair of disjoint subsets $B_1, B_2\subseteq\{1,\ldots,n\}$,

$$\sum_{\{\sigma\in B_n\mid \overleftarrow{\mathrm{Min}}_1(\sigma)=B_1,\overleftarrow{\mathrm{Min}}_0(\sigma)=B_2\}} q^{\ell(\sigma)}=\sum_{\{\sigma\in B_n\mid \overleftarrow{\mathrm{Min}}_1(\sigma)=B_1,\overleftarrow{\mathrm{Min}}_0(\sigma)=B_2\}} q^{fmaj(\sigma)}.$$

Proof: Combine Theorem 7.1 with Theorem 7.3. ∎

Corollary 7.6: *For every positive integer n*

$$\sum_{\sigma\in B_n} q^{\ell(\sigma)}x^{\overleftarrow{\mathrm{min}}_0(\sigma)}t^{\overleftarrow{\mathrm{min}}_1(\sigma)}=\sum_{\sigma\in B_n} q^{fmaj(\sigma)}x^{\overleftarrow{\mathrm{min}}_0(\sigma)}t^{\overleftarrow{\mathrm{min}}_1(\sigma)}$$
$$=(x+qt)(x+q+q^2+q^3t)\cdots(x+q+q^2+\cdots+q^{2n-1}t).$$

Proof: Substitute $x_1=\cdots=x_n=x$ and $t_1=\cdots=t_n=t$ in the r.h.s. of Theorems 7.1 and 7.3. ∎

8. Generalized binomial-Stirling numbers

In this section we present the generalized binomial-Stirling numbers, defined by a natural recurrence relation.

8.1 The Recurrence: Main Examples.

Definition 8.1: Fix three integers $a, d, r \in Z$ and let $h(n,k) = h_{a,d,r}(n,k)$ be the numbers determined by the following recurrence:

$$h(n,k) = (an + dk - r) \cdot h(n-1,k) + h(n-1,k-1), \tag{6}$$

where $h(0,0) = 1$ and $h(n,k) = 0$ if $k < 0$ or $n < k$.

We call $h_{a,d,r}(n,k)$ **the** (a,d,r)**-binomial-Stirling numbers**.

The following examples justify that terminology.

Example 8.2: The three main examples of such a system of numbers are the binomial coefficients and the two types of the Stirling numbers.

1. $a = d = 0$, $r = -1$, so $h(n,k) = h(n-1,k) + h(n-1,k-1)$. In this case $h(n,k) = \binom{n}{k}$ are the binomial coefficients.
2. $a = r = 1$, $d = 0$, so $h(n,k) = (n-1) \cdot h(n-1,k) + h(n-1,k-1)$. Thus $h(n,k) = c(n,k)$ are the signless Stirling numbers of the first kind.
3. $a = r = 0$, $d = 1$, hence $h(n,k) = k \cdot h(n-1,k) + h(n-1,k-1)$. Here $h(n,k) = S(n,k)$ are the Stirling numbers of the second kind.

8.2 Matrix Product Decomposition. We need to introduce some notation. Denote the (i,j)-th binomial coefficients by

$$b(i,j) := \binom{i}{j}.$$

Notation: We follow [17]. For $1 \le k \le n$, the signless Stirling numbers of the first kind are denoted by $c(n,k)$, $s(n,k) = (-1)^{n-k} c(n,k)$ are the Stirling numbers of the first kind, and $S(n,k)$ denote the Stirling numbers of the second kind.

Let $a, d, r \in Z$ and denote $r_1 = r + d$. Assume $a, d, r_1 \neq 0$. For the cases where some of these integers are zero, see Remark 8.5, Corollary 8.9 and Appendix 2 below.

For a positive integer n construct the following $n \times n$ lower-triangular matrices:

1. $c_n = (c(i,j) \mid 1 \le i,j \le n)$,
2. $s_n = (s(i,j) \mid 1 \le i,j \le n)$,
3. $S_n = (S(i,j) \mid 1 \le i,j \le n)$,
4. $P_n = (b(i,j) \mid 0 \le i,j \le n-1)$,
5. $J_n = \mathrm{diag}(1, -1, 1, -1, \ldots, (-1)^{n-1})$,
6. $a_n = \mathrm{diag}(1, a, a^2, \ldots, a^{n-1})$,

7. $d_n = \text{diag}(1, d, d^2, \ldots, d^{n-1})$,
8. $\hat{r}_n = \text{diag}(1, r_1^2, \ldots, r_1^{n-1})$, where $r_1 = r + d$.

The following properties are either obvious or well known.

LEMMA 8.3:
1. $J_n = J_n^{-1}$.
2. $P_n^{-1} = J_n P_n J_n = ((-1)^{i-j} b(i,j) \mid 0 \le i,j \le n-1)$.
3. $s_n = J_n c_n J_n$ *and* $S_n = s_n^{-1}$, *hence* $c_n = J_n S_n^{-1} J_n$.
4. $a_n s_n a_n^{-1} = (a^{i-j} s(i,j) \mid 1 \le i,j \le n)$,
   $d_n S_n d_n^{-1} = (d^{i-j} S(i,j) \mid 1 \le i,j \le n)$ *and*
   $\hat{r}_n P_n \hat{r}_n^{-1} = (r_1^{i-j} b(i,j) \mid 0 \le i,j \le n-1)$.
5. *The matrices* $a_n, d_n, \hat{r}_n$ *and* J_n *commute with each other.*
6. $\lim_{a\to 0} a_n s_n a_n^{-1} = \lim_{d\to 0} d_n S_n d_n^{-1} = \lim_{r_1\to 0} \hat{r}_n P_n \hat{r}_n^{-1} = I_n$, *where* I_n *is the* $n \times n$ *identity matrix.*

THEOREM 8.4: *Let* $a, d, r \in Z$, $r_1 = r + d$ *and* $a, d, r_1 \ne 0$. *Let* $h(n,k)$ *be a system of numbers such that the matrices* $h_n = (h(i,j) \mid 0 \le i,j \le n-1)$ *are lower triangular — for all* n.

Then $h(n,k)$ *satisfy the recurrence (6) if and only if the following matrix equations hold for all* n:

$$h_n = (a_n c_n a_n^{-1})(\hat{r}_n P_n^{-1} \hat{r}_n^{-1})(d_n S_n d_n^{-1}). \tag{7}$$

Proof: Let $\bar{h}(i,j)$ denote the entries on the right-hand side of (7). It suffices to show that the numbers $\bar{h}(n,k)$ satisfy the recurrence (6).

Since $\bar{h}(p,q)$ are given by the r.h.s. of (7), by matrix multiplication,

$$\bar{h}(p,q) = \sum_{p \ge j \ge i \ge q} a^{p-j}(-r_1)^{j-i} d^{i-q} c(p+1, j+1) \cdot b(j,i) \cdot S(i+1, q+1) \tag{8}$$

$$= \sum_{\infty \le i,j \le \infty} a^{p-j}(-r_1)^{j-i} d^{i-q} c(p+1, j+1) \cdot b(j,i) \cdot S(i+1, q+1). \tag{9}$$

The last equality follows from the defining conditions $c(n,k) = 0$ for $k < 0$ and $k > n$, and similarly for $b(n,k)$ and $S(n,k)$. Writing the sum in this form allows us to ignore the sum limits.

Clearly, $\bar{h}(0,0) = 1$. Apply now Equation (8) to show that the numbers $\bar{h}(n,k)$ satisfy the recurrence (6), namely, that

$$\bar{h}(n,k) = (an + dk - r) \cdot \bar{h}(n-1, k) + \bar{h}(n-1, k-1), \tag{10}$$

and this will prove the theorem.

By (8), since $r_1 = r + d$, the right-hand side of (10) is

$$(a(n-1)+d(k+1)+(a-r_1))\cdot\bar{h}(n-1,k)+\bar{h}(n-1,k-1) = M_1+M_2+M_3+M_4+M_5$$

where

$$M_1 = a(n-1)\cdot\bar{h}(n-1,k)$$
$$= \sum_{i,j} a^{n-j}(-r_1)^{j-i}d^{i-k}(n-1)\cdot c(n,j+1)\cdot b(j,i)\cdot S(i+1,k+1), \tag{11}$$

$$M_2 = d(k+1)\cdot\bar{h}(n-1,k)$$
$$= \sum_{i,j} a^{n-1-j}(-r_1)^{j-i}d^{i-k+1}c(n,j+1)\cdot b(j,i)\cdot(k+1)\cdot S(i+1,k+1), \tag{12}$$

$$M_3 = \bar{h}(n-1,k-1)$$
$$= \sum_{i,j} a^{n-1-j}(-r_1)^{j-i}d^{i-k+1}c(n,j+1)\cdot b(j,i)\cdot S(i+1,k), \tag{13}$$

$$M_4 = a\cdot\bar{h}(n-1,k)$$
$$= \sum_{i,j} a^{n-j}(-r_1)^{j-i}d^{i-k}c(n,j+1)\cdot b(j,i)\cdot S(i+1,k+1) \tag{14}$$

and

$$M_5 = (-r_1)\cdot\bar{h}(n-1,k)$$
$$= \sum_{i,j} a^{n-1-j}(-r_1)^{j+1-i}d^{i-k}c(n,j+1)\cdot b(j,i)\cdot S(i+1,k+1). \tag{15}$$

The recurrence $(k+1)S(i+1,k+1) + S(i+1,k) = S(i+2,k+1)$ implies that

$$M_2 + M_3 = \sum_{i,j} a^{n-1-j}(-r_1)^{j-i}d^{i-k+1}c(n,j+1)\cdot b(j,i)\cdot S(i+2,k+1). \tag{16}$$

Replacing $i+1$ by i and $j+1$ by j, Equation (16) implies that

$$M_2 + M_3 = \sum_{i,j} a^{n-j}(-r_1)^{j-i}d^{i-k}c(n,j)\cdot b(j-1,i-1)\cdot S(i+1,k+1). \tag{17}$$

Replacing $j+1$ by j in (15) yields

$$M_5 = \sum_{i,j} a^{n-j}(-r_1)^{j-i}d^{i-k}c(n,j)\cdot b(j-1,i)\cdot S(i+1,k+1). \tag{18}$$

Since $b(j-1,i) + b(j-1,i-1) = b(j,i)$, by (17) and (18)

$$M_2 + M_3 + M_5 = \sum_{i,j} a^{n-j}(-r_1)^{j-i}d^{i-k}c(n,j)\cdot b(j,i)\cdot S(i+1,k+1). \tag{19}$$

Clearly

$$(20)\quad M_1 + M_4 = \sum_{i,j} a^{n-j}(-r_1)^{j-i}d^{i-k}n \cdot c(n,j+1)\cdot b(j,i)\cdot S(i+1,k+1).$$

Since $n \cdot c(n,j+1) + c(n,j) = c(n+1,j+1)$, by (19), (20) and (8) we finally obtain

$$M_1 + M_2 + M_3 + M_4 + M_5$$
$$= \sum_{i,j} a^{n-j}(-r_1)^{j-i}d^{i-k}c(n+1,j+1)\cdot b(j,i)\cdot S(i+1,k+1) = \bar{h}(n,k).$$

This completes the proof. ∎

Remark 8.5: Theorem 8.4 holds when $a = 0$: in that case, the factor $a_n c_n a_n^{-1}$ is canceled from (7) since $\lim_{a\to 0} a_n s_n a_n^{-1} = I_n$. Similarly, it holds when $d = 0$ and when $r_1 = 0$.

Remark 8.6: If one reverses the order in (7), it seems that the numbers given by

$$h_n^* = (d_n S_n d_n^{-1})(\hat{r}_n P_n^{-1}\hat{r}_n^{-1})(a_n c_n a_n^{-1})$$

satisfy no (obvious) recurrence.

8.3 THE DUAL SYSTEM $Q_{a,d,r,}(n,k)$. By a trivial induction on i, $h_{a,d,r}(i,i) = 1$ for all i. Also, by definition, the matrix $((-1)^{i-j}h_{a,d,r}(i,j))_{0\le i,j\le n-1}$ is lower triangular, hence invertible. By inversion we obtain the **dual system** $Q_{a,d,r}(n,k)$:

$$(Q_{a,d,r}(i,j))_{0\le i,j\le n-1} := ((-1)^{i-j}h_{a,d,r}(i,j))^{-1}_{0\le i,j\le n-1},$$

and $Q_{a,d,r}(n,k)$ might be called the (a,d,r)-binomial-Stirling numbers of the second kind.

Definition 8.7: Again, let $a, d, r \in Z$ and define $h(n,k) = h_{a,d,r}(n,k)$ via either (6) or (7).

1. Call $h(n,k)$ the (a,d,r)-signless binomial-Stirling numbers of the first kind. Let
$$q(n,k) = (-1)^{n-k}h(n,k),$$
and call $q(n,k)$ the (a,d,r)-binomial-Stirling numbers of the first kind. Finally, denote
$$h_n = (h(i,j) \mid 0 \le i,j \le n-1) \quad \text{and} \quad q_n = (q(i,j) \mid 0 \le i,j \le n-1)$$

$n \times n$ lower-triangular matrices.

2. Define the numbers $Q(i,j)$ by inverting the matrices q_n:

$$Q_n = (Q(i,j) \mid 0 \le i,j \le n-1) = q_n^{-1} = (q(i,j) \mid 0 \le i,j \le n-1)^{-1}$$
$$= ((-1)^{i-j} h(i,j) \mid 0 \le i,j \le n-1)^{-1}.$$

Also, $Q(i,j) = 0$ if $j < 0$ or if $i < j$. The definition of $Q(i,j)$ is independent of n, provided $i \le n$.

Call $Q(n,k) = Q_{a,d,r}(n,k)$ the (a,d,r)-binomial-Stirling numbers of the second kind.

The following theorem shows that such binomial-Stirling numbers of the second kind are just a binomial-Stirling system, but with $(d, a, r+d-a)$ replacing (a,d,r).

Clearly, $q_n = J_n h_n J_n$, hence $h_n = J_n Q_n^{-1} J_n$.

With the notation of Subsection 8.2 we have

THEOREM 8.8: *Let $Q(n,k) = Q_{a,d,r}(n,k)$ denote the (a,d,r)-Stirling numbers of the second kind, with corresponding matrices Q_n. Then*

1.

$$Q_n = (d_n c_n d_n^{-1})(\hat{r}_n P_n^{-1} \hat{r}_n^{-1})(a_n S_n a_n^{-1}). \tag{21}$$

2. *The numbers $Q(n,k) = Q_{a,d,r}(n,k)$ satisfy the following recurrence, which is "dual" to the recurrence (6):*

$$Q(n,k) = (dn + ak - r_2) \cdot Q(n-1,k) + Q(n-1,k-1), \tag{22}$$

where $r_2 = r + d - a$. Thus $Q(n,k) = Q_{a,d,r}(n,k) = h_{d,a,r+d-a}(n,k)$.

Proof: 1. By Lemma 8.3.1 and Definition 8.7, $h_n^{-1} = J_n Q_n J_n$. Inverting (7) implies that

$$J_n Q_n J_n = (d_n S_n^{-1} d_n^{-1})(\hat{r}_n P_n \hat{r}_n^{-1})(a_n c_n^{-1} a_n^{-1}).$$

Applying Lemma 8.3, deduce that

$$Q_n = J_n (d_n J_n c_n J_n d_n^{-1})(\hat{r}_n P_n \hat{r}_n^{-1})(a_n J_n S_n J_n a_n^{-1}) J_n$$
$$= (d_n c_n d_n^{-1})(\hat{r}_n J_n P_n J_n \hat{r}_n^{-1})(a_n S_n a_n^{-1}),$$

and the proof follows by Lemma 8.3.2.

2. By Theorem 8.4, (22) and (21) are equivalent, with $r_1 = r_2 + a$. ∎

COROLLARY 8.9: *In particular, the two systems of* $h_{a,0,1}(n,k)$ *and of* $h_{0,a,1-a}(n,k)$ *are dual to each other: the system of* $h_{0,a,1-a}(n,k)$ *is obtained by inverting the corresponding lower-triangular matrix with entries* $(-1)^{n-k}h_{a,0,1}(n,k)$.

Remark 8.10: Theorem 8.8 holds when either of a, d or $r_1 = r + d$ equals zero. For example, when $a = 0$ apply $\lim_{a\to 0}$ to (21), using Lemma 8.3.6; see Remark 8.5. Similarly for $d = 0$ or $r_1 = 0$.

8.4 THE (a,d,r,ℓ) SYSTEMS. Let $\ell = \ell' \neq 0$ and consider the system of numbers $g(n,k) = g_{a',d',r',\ell'}(n,k)$ given by the following (a',d',r',ℓ')-recurrence:

Again $g(0,0) = 1$, and

$$g(n,k) = (a'n + d'k - r')\cdot g(n-1,k) + \ell'\cdot g(n-1,k-1). \tag{23}$$

By a trivial induction one proves:

Remark 8.11: Let $a = a'/\ell$, $d = d'/\ell$, $r = r'/\ell$, $\ell' = \ell$, and let $h(n,k) = h_{a,d,r}(n,k)$ be given as in Equation (6). Then for all n and k,

$$g_{a',d',r',\ell'}(n,k) = \ell^n \cdot h_{a,d,r}(n,k). \tag{24}$$

Thus

$$g_{a,d,r,1}(n,k) = h_{a,d,r}(n,k). \tag{25}$$

Similar to the dual system $Q(n,k) = Q_{a,d,r}(n,k)$ of $h_{a,d,r}(n,k)$, construct the dual system $V(n,k) = V_{a',d',r',\ell'}(n,k)$ of $g_{a',d',r',\ell'}(n,k)$ as follows:

Let $v(n,k) = (-1)^{n-k}g(n,k)$, $v_n = [v(i,j) \mid 1 \le i,j \le n]$ and the numbers $V(n,k) = V_{a',d',r',\ell'}(n,k)$ are given by the matrix equation

$$v_n^{-1} = [V(i,j) \mid 1 \le i,j \le n].$$

By matrix inversion one proves

Remark 8.12: For all n and k,

$$V_{a',d',r',\ell'}(n,k) = \ell^{-k}Q_{a,d,r}(n,k). \tag{26}$$

9. Realizations of the dual systems

In Sections 5 and 6 two systems of binomial-Stirling numbers are realized by certain statistics on colored permutations. It is shown here that these two systems are dual to each other — in the sense of Section 8.

Remark 9.1: 1. Note that Corollary 5.2 can be considered as a "wreath-product-realization" of the system $g_{a,0,\ell,\ell}(n,k)$ with $0 \leq \ell \leq a-1$: the recurrence of $g_L(n,k)$ there implies that $g_L(n,k) = g_{a,0,\ell,\ell}(n,k)$, thus

$$g_{a,0,\ell,\ell}(n,k) = \#\{\sigma \in C_a \wr S_n \; | \overleftarrow{\min}_L (\sigma) = k\}.$$

In particular, if $L = \{u\}$ then $\ell = 1$, $g_{a,0,1,1}(n,k) = h_{a,0,1}(n,k)$ and we have

$$h_{a,0,1}(n,k) = \#\{\sigma \in C_a \wr S_n \; | \overleftarrow{\min}_u (\sigma) = k\}.$$

2. Similarly, Theorem 6.6 (with d replacing a) is a"wreath-product-realization" of the system $g_{0,d,\ell-d,\ell}(n,k)$ with $0 \leq \ell \leq d-1$: the recurrence of $f_L(n,k)$ there implies that $f_L(n,k) = g_{0,d,\ell-d,\ell}(n,k)$, and by Definition 6.5

$$g_{0,d,\ell-d,\ell}(n,k) = \#\{\sigma \in C_d \wr S_n \mid des_L(\sigma) = \overleftarrow{\min}_L (\sigma) = k\}.$$

In particular, if $L = \{u\}$ then $\ell = 1$, $g_{0,d,\ell-d,\ell}(n,k) = h_{0,d,1-d}(n,k)$, hence

$$h_{0,d,1-d}(n,k) = \#\{\sigma \in C_d \wr S_n \mid des_u(\sigma) = \overleftarrow{\min}_u (\sigma) = k\}.$$

3. Summing the above on k implies

$$\sum_k g_{0,d,\ell-d,\ell}(n,k) = \#\{\sigma \in C_d \wr S_n \mid des_L(\sigma) = \overleftarrow{\min}_L (\sigma)\}.$$

This leads to the (d,r)-**Bell numbers** and with the following wreath-product-realization:

Definition 9.2: Recall the numbers $h_{0,d,r}(n,k) = g_{0,d,r,1}(n,k)$, denote

$$b_{d,r}(n) = \sum_k h_{0,d,r}(n,k) = \sum_k g_{0,d,r,1}(n,k),$$

and call these the (d,r)-Bell numbers.

Note that by Example 8.2.3, $h_{0,1,0}(n,k) = S(n,k)$ are the Stirling numbers of the second kind, therefore $b_{1,0}(n)$ are the (ordinary) Bell-numbers. Further properties of the (d,r)-Bell numbers are given in Appendix 2.

By Remark 9.1.3 and the above definition,

COROLLARY 9.3:

$$b_{d,1-d}(n) = \#\{\sigma \in C_d \wr S_n |\ des_u(\sigma) = \overleftarrow{\min}_u (\sigma)\}.$$

Recall from [16, Propositions 10.8 and 10.10] that the signed Stirling number of the first kind, $s(n,k)$, is equal to

$$(-1)^{n-k} \cdot \#\{\pi \in S_n |\ \overleftarrow{\min} (\pi) = k\},$$

while the Stirling number of the second kind, $S(n,k)$, is equal to

$$\#\{\pi \in S_n |\ des(\pi) = \overleftarrow{\min} (\pi) = k\}.$$

These numbers form inverse matrices; see, e.g.,[17, Prop. 1.4.1.a]. This phenomenon is generalized to wreath products.

THEOREM 9.4: *For every positive integers a, N, and every subset $L \subseteq \{0,\ldots,a-1\}$ of size ℓ, let $s_{L,N}$ be the $N \times N$ matrix whose entries are given by*

$$s_{L,N}(n,k) := \frac{(-1)^{n-k}}{\ell^n} \cdot \#\{\sigma \in C_a \wr S_n |\ \overleftarrow{\min}_L (\sigma) = k\} \quad (0 \le k, n \le N)$$

and $S_{L,N}$ be the $N \times N$ matrix whose entries are given by

$$S_{L,N}(n,k) := \frac{1}{\ell^n} \cdot \#\{\sigma \in C_a \wr S_n |\ des_u(\sigma) = \overleftarrow{\min}_u (\sigma) = k\} \quad (0 \le k, n \le N).$$

Then

$$S_{L,N}^{-1} = s_{L,N}.$$

Proof: First note that the results in Section 8 hold for any rational (essentially real) a, d, r. Thus, by Remarks 9.1.(1) and 8.11,

$$\ell^{-n} \cdot \#\{\sigma \in C_a \wr S_n\ |\overleftarrow{\min}_L (\sigma) = k\} = \ell^{-n} g_{a,0,\ell,\ell}(n,k) = h_{a/\ell,0,1}(n,k).$$

Similarly by Remarks 9.1.2 and 8.11,

$$\begin{aligned} &\ell^{-n} \cdot \#\{\sigma \in C_a \wr S_n \mid des_L(\sigma) = \overleftarrow{\min}_L (\sigma) = k\} \\ &= \ell^{-n} g_{0,a,\ell-a,\ell}(n,k) = h_{0,a/\ell,1-a/\ell}(n,k). \end{aligned}$$

Corollary 8.9 completes the proof. ∎

10. Appendix 1: Proof of Proposition 4.9

For every element $\sigma \in C_a \wr S_n$ and $L \subseteq \{0, \dots, a-1\}$ define

$$\mathrm{inv}_L(\sigma) := \{(i,j) \mid i < j \quad \text{and} \quad \sigma(i) >_L \sigma(j)\}.$$

For $1 \leq i \leq n$ denote the color of $\sigma(i)$ by $z_i(\sigma)$. Namely, $z_i(\sigma) = j$ if $\sigma(i) = \alpha^j |\sigma(i)|$.

We will apply the following combinatorial formula for the length function.

LEMMA 10.1 ([4, Theorem 3.3.3]): *For every positive integers a and n, and every element $\sigma \in C_a \wr S_n$,*

$$\ell(\sigma) = \mathrm{inv}_{\bar{0}}(\sigma) + \sum_{\sigma(i) <_{\bar{0}} 0} (|\sigma(i)| - 1) + \sum_{j=1}^{n} z_j(\sigma),$$

where $\bar{0} := \{1, \dots, a-1\}$.

COROLLARY 10.2: *For every element $\sigma \in C_a \wr S_n$ and $0 \leq i \leq n-1$,*

$$\ell(\sigma s_i) < \ell(\sigma) \Longleftrightarrow \sigma(i) >_{\bar{0}} \sigma(i+1)$$

where we assume $\sigma(0) = 0$ and $\bar{0} := \{1, \dots, a-1\}$.

Proof: By the definition of the order $<_{\bar{0}}$ together with Fact 2.1(1) the corollary holds for $i = 0$. For $i \neq 0$, the corollary follows from Lemma 10.1 together with Fact 2.1(2). ∎

Proof of Proposition 4.9: By Corollary 10.2,

$$\{0 \leq i \leq n-1 \mid \ell(v_u^{-1}\sigma s_i) < \ell(v_u^{-1}\sigma)\} = \{0 \leq i \leq n-1 \mid v_u^{-1}\sigma(i) >_{\bar{0}} v_u^{-1}\sigma(i+1)\}.$$

One may easily verify that

$$v_u^{-1}\sigma(i) >_{\bar{0}} v_u^{-1}\sigma(i+1) \Longleftrightarrow \sigma(i) >_{\bar{u}} \sigma(i+1),$$

where $\bar{u} := \{0, \dots, a-1\} \setminus u$.

This proves that for every $0 \leq u \leq a-1$,

$$Des_{\bar{u}}(\sigma) = \{0 \leq i \leq n-1 \mid \ell(v_u^{-1}\sigma s_i) < \ell(v_u^{-1}\sigma)\}. \tag{27}$$

Proposition 4.9 is deduced from (27) by observing that $>_{\bar{u}}$ is the reverse order of $>_u$; hence $Des_u(\sigma) = \{0, \dots, n-1\} \setminus Des_{\bar{u}}$. ∎

11. Appendix 2: Further properties of the generalized binomial-Stirling and Bell numbers

In this appendix we study some further properties of the generalized binomial-Stirling and Bell numbers, introduced in Section 8.

PROPOSITION 11.1: *Let $d = 0$, namely, the numbers $h_{a,0,r}(n,k)$ satisfy the recurrence (6) with $d = 0$. Let*

$$f_n(x) := \sum_k h_{a,0,r}(n,k)x^k. \tag{28}$$

Then

$$f_n(x) = (x + a - r)(x + 2a - r) \cdots (x + na - r). \tag{29}$$

In particular, $\sum_k h_{a,0,r}(n,k) = (a - r + 1)(2a - r + 1) \cdots (na - r + 1)$.

Proof: For $n \geq 1$ let $\bar{h}(n,k)$ be the coefficient of x^k in the following expansion:

$$\bar{f}_n(x) = (a - r + x)(2a - r + x) \cdots (na - r + x) = \sum_k \bar{h}(n,k)x^k,$$

and define $\bar{h}(0,0) := 1$.

Then $\bar{f}_n(x) = (x + na - r)\bar{f}_{n-1}(x) = (na - r)\bar{f}_{n-1}(x) + x\bar{f}_{n-1}(x)$. It easily follows that $\bar{h}(n,k)$ satisfies the same recurrence as $h_{a,0,r}(n,k)$, which implies that $\bar{h}(n,k) = h_{a,0,r}(n,k)$. ∎

In the rest of this section we study the binomial-Stirling numbers with $a = 0$, namely $h(n,k) = h_{0,d,r}(n,k)$, and deduce further results about these numbers and their sums, the (d,r)-Bell numbers. We follow closely Section 1.6 of [19].

Denote

$$g_k(y) := \sum_n h(n,k)y^n = \sum_n h_{0,d,r}(n,k)y^n. \tag{30}$$

Thus $h_{0,d,r}(n,k)$ is the coefficient of y^n in $g_k(y)$.

PROPOSITION 11.2: *Let $a = 0$, namely, the numbers $h(n,k) = h_{0,d,r}(n,k)$ satisfy recurrence (6) with $a = 0$:*

$$h(n,k) = (dk - r) \cdot h(n-1,k) + h(n-1,k-1). \tag{31}$$

Then

$$g_k(y) = \frac{y^k}{(1-(-r)y)(1-(d-r)y)\cdots(1-(kd-r)y)}. \tag{32}$$

Proof: Define $\bar{g}_k(y)$ and $\bar{h}(n,k)$ via the expansion of the following ratio:

$$\bar{g}_k(y) = \frac{y^k}{(1-(-r)y)(1-(d-r)y)\cdots(1-(kd-r)y)} = \sum_n \bar{h}(n,k)y^n. \tag{33}$$

Clearly

$$\bar{g}_k(y) = \frac{y}{1-(dk-r)y} \cdot \bar{g}_{k-1}(y),$$

hence $\bar{g}_k(y) = (dk-r)\cdot y \cdot \bar{g}_k(y) + y \cdot \bar{g}_{k-1}(y)$, namely,

$$\sum_n \bar{h}(n,k)y^n = \sum_n (dk-r)\bar{h}(n-1,k)y^n + \sum_n \bar{h}(n-1,k-1)y^n.$$

Comparing coefficients, it follows that $\bar{h}(n,k)$ satisfies the same recurrence (31) as $h(n,k)$, hence $h(n,k) = \bar{h}(n,k)$, which completes the proof. ∎

COROLLARY 11.3 (see [17, Ex. 16 in Ch. 1]):

$$h_{0,d,r}(n,k) = \sum (-r)^{a_0-1}\cdot(d-r)^{a_1-1}\cdots(kd-r)^{a_k-1}, \tag{34}$$

the sum being over all compositions $a_1+\cdots+a_{k+1} = n+1$ *where all* $a_i \geq 1$.

It should be interesting to give Equation (34) a purely combinatorial proof. The following proposition extends [19, (1.6.7)].

PROPOSITION 11.4:

$$h_{0,d,r}(n,k) = \sum_{t=0}^{k} (-1)^{k-t}\cdot\frac{(td-r)^n}{d^k\cdot t!(k-t)!}. \tag{35}$$

Proof: Let

$$g_k^*(y) = y^{-k}g_k(y) = \frac{1}{(1-(-r)y)(1-(d-r)y)\cdots(1-(kd-r)y)},$$

and notice that $h_{0,d,r}(n,k)$ is the coefficient of y^{n-k} in $g_k^*(y)$. Applying partial fractions, this can be written as

$$\frac{1}{(1-(-r)y)(1-(d-r)y)\cdots(1-(kd-r)y)} = \sum_{t=0}^{k}\frac{\alpha_t}{1-(td-r)y}$$

with some $\alpha_t \in R$.

To calculate α_t, multiply both sides by $1-(td-r)y$, then substitute $y = 1/(td-r)$. On the right we get α_t and on the left

$$\frac{1}{(1-(-r)y)\cdots(1-((t-1)d-r)y)(1-((t+1)d-r)y)\cdots(1-(kd-r)y)}$$
$$= \frac{1}{(1-\frac{0\cdot d-r}{t\cdot d-r})\cdots(1-\frac{(t-1)\cdot d-r}{t\cdot d-r})\cdot(1-\frac{(t+1)\cdot d-r}{t\cdot d-r})\cdot(1-\frac{k\cdot d-r}{t\cdot d-r})}$$
$$= \frac{(td-r)^k}{td\cdot(t-1)d\cdots d\cdot(-d)\cdot(-2d)\cdots(-(k-t)d)}$$
$$= (-1)^{k-t}\frac{(td-r)^k}{d^k\cdot t!(k-t)!}.$$

Deduce that

$$\alpha_t = (-1)^{k-t}\frac{(td-r)^k}{d^k\cdot t!(k-t)!}.$$

Recall that $h(n,k)$ is the coefficient of y^{n-k} in $g_k^*(y)$, and that

$$g_k^*(y) = \sum_{t=0}^{k}\frac{\alpha_t}{1-((td-r)y}.$$

Thus

$$h(n,k) = [y^{n-k}]\sum_{t=0}^{k}\frac{\alpha_t}{1-(td-r)y}$$
$$= \sum_{t=0}^{k}[y^{n-k}]\frac{\alpha_t}{1-(td-r)y} = \sum_{t=0}^{k}(td-r)^{n-k}\alpha_t$$
$$= \sum_{t=0}^{k}(td-r)^{n-k}(-1)^{k-t}\cdot\frac{(td-r)^k}{d^k\cdot t!(k-t)!}$$
$$= \sum_{t=0}^{k}(-1)^{k-t}\cdot\frac{(td-r)^n}{d^k\cdot t!(k-t)!}. \qquad \blacksquare$$

Recall the (d,r)-Bell numbers $b_{d,r}(n) = \sum_k h_{0,d,r}(n,k)$ from Definition 9.2. We have the following formula for these numbers, extending a remarkable result of Dobinski [6].

Proposition 11.5:

$$b_{d,r}(n) = \frac{1}{e^{1/d}}\sum_{t=0}^{\infty}\frac{(td-r)^n}{t!d^t}.$$

Proof: We continue to follow [19].

Choose M large enough. Then, by the previous proposition,

$$\begin{aligned} b_{d,r}(n) &= \sum_{k=0}^{M}\sum_{t=0}^{k}(-1)^{k-t}\cdot\frac{(td-r)^n}{d^k\cdot t!(k-t)!} \\ &= \sum_{t=0}^{M}\frac{(td-r)^n}{t!d^t}\cdot\sum_{k=t}^{M}\frac{(-1)^{k-t}}{(k-t)!d^{k-t}} \\ &= \sum_{t=0}^{M}\frac{(td-r)^n}{t!d^t}\cdot\sum_{s=0}^{M-t}\frac{(-1)^s}{(s)!}\cdot\Big(\frac{1}{d}\Big)^s. \end{aligned}$$

The proof now follows by sending M to infinity, since then the second factor becomes

$$\sum_{s=0}^{\infty}\frac{(-1)^s}{(s)!}\cdot\Big(\frac{1}{d}\Big)^s = \frac{1}{e^{1/d}}.$$ ∎

COROLLARY 11.6: *For every positive n,*

$$\#\{\sigma \in B_n \mid des(\sigma) = \overleftarrow{\min}_1(\sigma)\} = \frac{1}{\sqrt{e}}\sum_{t=0}^{\infty}\frac{(2t+1)^n}{t!2^t}, \tag{36}$$

where $des(\sigma) = \#\{0 \le i \le n-1 \mid \ell(\sigma s_i) < \ell(\sigma)\}$ is the standard descent number.

Proof: Combine Corollary 9.3 with Proposition 11.5 (letting $d = 2$ and $r = -1$). The identity $des(\sigma) = des_1(\sigma)$ ($\forall\sigma \in B_n$) (see Example 4.10.3) completes the proof. ∎

Definition 11.7: Let $B_{d,r}(x)$ be the exponential generating function of the $b_{d,r}(n)$'s:

$$B_{d,r}(x) = \sum_{n=0}^{\infty} b_{d,r}(n)\frac{x^n}{n!}.$$

PROPOSITION 11.8:

$$B_{d,r}(x) = \exp\Big(\frac{e^{dx} - drx - 1}{d}\Big).$$

Proof: By definition, $b_{d,r}(0) = 1$; hence, by Proposition 11.5,

$$\begin{aligned} B_{d,r}(x) - 1 &= \frac{1}{e^{1/d}} \sum_{n=1}^{\infty} \frac{x^n}{n!} \sum_{t=0}^{\infty} \frac{(td-r)^n}{t!d^t} \\ &= \frac{1}{e^{1/d}} \sum_{t=0}^{\infty} \frac{1}{t!d^t} \sum_{n=1}^{\infty} \frac{[(td-r)x]^n}{n!} \\ &= \frac{1}{e^{1/d}} \sum_{t=0}^{\infty} \frac{1}{t!d^t} \cdot (e^{(td-r)x} - 1) \\ &= -1 + \frac{e^{-rx}}{e^{1/d}} \sum_{t=0}^{\infty} \frac{1}{t!d^t} \cdot e^{tdx}. \end{aligned}$$

Thus

$$\begin{aligned} B_{d,r}(x) &= e^{-(drx+1)/d} \sum_{t=0}^{\infty} \frac{1}{t!} \cdot \left(\frac{e^{dx}}{d}\right)^t \\ &= e^{-(drx+1)/d} \cdot e^{(e^{dx}/d)} = \exp\left(\frac{e^{dx} - drx - 1}{d}\right). \end{aligned}$$ ∎

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